

On the zone complexity of a vertex

Shira Zerbib*

April 26, 2016

Abstract

Let L be a set of n lines in the real projective plane in general position. We show that there exists a vertex $v \in \mathcal{A}(L)$ such that v is positioned in a face of size at most 5 in the arrangement obtained by removing the two lines passing through v .

1 Introduction

Let L be a set of n lines in the real projective plane in general position (i.e., no three of which pass through the same point), and let $\mathcal{A}(L)$ be the planar arrangement of vertices, edges and faces determined by L . The *zone* $Z(\ell)$ of a line $\ell \in L$ is defined as the collection of faces supported by ℓ . The *complexity* $C(\ell)$ of the zone of ℓ is the sum of the sizes of all the faces in $\mathcal{A}(L \setminus \{\ell\})$ which contain a segment of ℓ . The zone theorem asserts that $C(\ell) \leq 5.5(n - 1) + O(1)$, and that this bound is tight [2]. This was an improvement of the upper bound of $6(n - 1)$ given in [3] and [4].

A natural and interesting question that arises is whether the bound given by the zone theorem can be improved on average. Namely, we ask:

Question 1. *Let L be a set of n lines in the real projective plane in general position. Is it true that $\frac{1}{n} \sum_{\ell \in L} C(\ell) \leq c(n - 1)$, where c is a positive constant strictly smaller than 5.5?*

Looking for supporting evidence to a positive answer to Question 1, we were naturally led to define the notion of the zone complexity of a vertex. For a vertex $v \in \mathcal{A}(L)$, let ℓ_1^v, ℓ_2^v be the two lines passing through v . We define the *zone* $Z(v)$ of v as the collection of four faces containing v . The *complexity* $C(v)$ of the zone of v is defined as the size of the face in the arrangement $\mathcal{A}(L \setminus \{\ell_1^v, \ell_2^v\})$ which contains the position of v in the plane. Let $C(L)$ denote the minimal vertex zone complexity in $\mathcal{A}(L)$.

An easy consequence of the zone theorem is that every line $\ell \in L$ passes through a vertex v with $C(v) \leq 7$ (see Section 2 for a proof of this claim). This shows in particular that $C(L) \leq 7$ for any arrangement L . Now, if the answer to Question 1 is positive, it will similarly indicate that there must be a vertex $v \in \mathcal{A}(L)$ such that $C(v) < 7$, i.e., $C(L) < 7$. Our main result shows that this is indeed the case, namely we prove the following:

Theorem 1. *For every set L of n lines in the real projective plane in general position, $C(L) \leq 5$.*

*Department of Mathematics, Technion—Israel Institute of Technology, Haifa 32000, Israel. zgshira@tx.technion.ac.il.

Recall the standard representation of the real projective plane as a quotient space of the sphere, where any two antipodal points are identified. In this model, lines are represented by great circles on the sphere, and the crossing point of two lines is the pair of antipodal points at which the two respective great circles meet. The notions of the zone of a vertex and its complexity are translated on the sphere in an obvious way. Thus for a set L of n great circles on the sphere in general position we can define the minimal vertex complexity $C(L)$ as above. Translating Theorem 1, we can state our main result in the following equivalent way:

Theorem 2. *For every set L of n great circles on the sphere in general position, $C(L) \leq 5$.*

Theorem 2 is proved in Section 3. Moreover, in Section 4 we give an example of a set of lines L in the real projective plane for which every vertex $v \in \mathcal{A}(L)$ has $C(v) \geq 5$, which shows that the bound given in Theorem 1 is tight. Note that since the zone theorem gives a tight upper bound on the line zone complexity, the upper bound given in Theorem 1 cannot be achieved using the zone theorem by considering each line individually. Therefore Theorem 1 may suggest that the answer to Question 1 is positive.

2 An upper bound on $C(L)$ using the zone theorem

The notion of the zone complexity of a vertex is closely related to that of the zone complexity of a line. In fact, the zone complexity $C(\ell)$ of a line ℓ can be expressed in terms of the zone complexities of the vertices that lie on ℓ . In the following proposition we calculate this relation and prove, using the zone theorem, that $C(L) \leq 7$.

Proposition 1. *For every set L of n lines in the real projective plane in general position, $C(L) \leq 7$.*

Proof. We write $v \in \ell$ if v lies on ℓ (that is, if ℓ passes through v), and let $|f|$ be the size of a face f . The relation between the sizes of the faces in $Z(\ell)$ and the sizes of the faces in the zones of the vertices that lie on ℓ is given in the following equation:

$$\sum_{v \in \ell} \sum_{f \in Z(v)} |f| = 2 \sum_{f \in Z(\ell)} |f|. \quad (1)$$

Observe that for every vertex $v \in \mathcal{A}(L)$,

$$\sum_{f \in Z(v)} |f| = C(v) + 12. \quad (2)$$

Indeed, since v contributes 4 to $\sum_{f \in Z(v)} |f|$ and each of the 4 vertices adjacent to v contributes 2 to $\sum_{f \in Z(v)} |f|$, v and its neighbors contribute 12.

Now, since the number of vertices that lie on ℓ is $n - 1$, (2) implies

$$\sum_{v \in \ell} \sum_{f \in Z(v)} |f| = \sum_{v \in \ell} C(v) + 12(n - 1). \quad (3)$$

On the other hand, it is easy to see that

$$\sum_{f \in Z(\ell)} |f| = C(\ell) + 4(n - 1). \quad (4)$$

Therefore, combining (1), (3) and (4) we get the following relation between the zone complexity of a line ℓ and the zone complexities of the vertices that lie on it:

$$C(\ell) = \frac{1}{2} \sum_{v \in \ell} C(v) + 2(n-1). \quad (5)$$

Finally, set $r(\ell) := \min_{v \in \ell} C(v)$. From (5) and the zone theorem we get that

$$\frac{1}{2}(n-1) \cdot r(\ell) + 2(n-1) \leq C(\ell) \leq 5.5(n-1) + O(1). \quad (6)$$

Since the constant $O(1)$ in the zone theorem is actually negative (it equals -1 ; see the proof of Theorem 1 in [2]), it follows from (6) that $r(\ell) \leq 7$ for all n . Hence every line in L passes through a vertex v with $C(v) \leq 7$, and in particular $C(L) \leq 7$, as claimed. \square

3 Proof of Theorem 2

Throughout the proof we use the term *4-multiset* to describe a multiset of cardinality 4. We begin by proving an elementary technical lemma which will play a crucial role in the sequel.

Lemma 1. *Let $K = \{k_1, k_2, k_3, k_4\}$ be a 4-multiset of integers with the following properties:*

1. $k_i \geq 3$ for $i = 1, 2, 3, 4$.
2. At most two of the elements in K equal 3.
3. $\sum_{i=1}^4 k_i \geq 18$.

Then $\sum_{i=1}^4 \frac{k_i-3}{k_i} < 1$ if and only if K is one of the following 4-multisets:

$$\{3, 3, 4, 8\}, \{3, 3, 4, 9\}, \{3, 3, 4, 10\}, \{3, 3, 4, 11\}, \{3, 3, 5, 7\}.$$

Proof. Let $K = \{k_1, k_2, k_3, k_4\}$ be a 4-multiset with the above properties. Then $\sum_{i=1}^4 \frac{k_i-3}{k_i} < 1$ if and only if $\sum_{i=1}^4 \frac{1}{k_i} > 1$. Change the order in K such that $k_1 \leq k_2 \leq k_3 \leq k_4$. It follows from Property (2) that $k_3 \geq 4$. Moreover, $k_4 \geq 6$ or $k_3 \geq 5$, since otherwise $\sum_{i=1}^4 k_i \leq 4 + 4 + 4 + 5 = 17$ in contradiction to Property (3). If $k_2 \geq 4$ we have

$$\sum_{i=1}^4 \frac{1}{k_i} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} = 1.$$

Thus $\sum_{i=1}^4 \frac{1}{k_i} > 1$ implies $k_1 = k_2 = 3$. It is now straightforward to verify that there are only 5 possibilities for $\{k_3, k_4\}$: $\{4, 8\}$, $\{4, 9\}$, $\{4, 10\}$, $\{4, 11\}$ or $\{5, 7\}$, as claimed. \square

Let L be a set of n great circles in general position on the sphere. In order to prove that $C(L) \leq 5$ we need to show that there exists a vertex $v \in \mathcal{A}(L)$ such that $C(v) \leq 5$. Assume to the contrary that every vertex $v \in \mathcal{A}(L)$ has $C(v) \geq 6$. By (2), this assumption is equivalent to

$$\sum_{f \in Z(v)} |f| \geq 18 \quad (7)$$

for every v .

Denote by V the number of vertices, by E the number of edges, and by F the number of faces in the planar arrangement $\mathcal{A}(L)$. By Euler's formula, we have $V - E + F = 2$. For every $k \geq 3$ denote by f_k the number of faces in $\mathcal{A}(L)$ of size k . We observe that $4V = 2E = \sum k f_k$ and $F = \sum f_k$. Therefore,

$$-6 = -3V + E + 2E - 3F = -3V + 2V + \sum k f_k - 3 \sum f_k = -V + \sum (k - 3) f_k. \quad (8)$$

We are going now to use the discharging method. The discharging method is a technique often used to prove statements in structural graph theory, and is commonly applied in the context of planar graphs (for a review on the discharging method and some of its applications see [5], [1]). Our plan is to assign to every face and vertex of the arrangement $\mathcal{A}(L)$ an initial charge, such that the sum of all assigned charges is negative. Then we redistribute (discharge) the charges in two steps, such that after these two steps every face and vertex in $\mathcal{A}(L)$ will have a nonnegative charge. It will thus follow that the total initial charge is nonnegative, which is a contradiction.

Step 1 [initial charging]: We begin by assigning a charge $w_1(\cdot)$ to the faces and vertices of the arrangement $\mathcal{A}(L)$: The charge of a face of size k is $k - 3$, while the charge of each vertex is -1 . It follows from (8) that the overall charge is -6 .

Step 2 [discharging the faces]: For every $k \geq 3$, every face f of size k contributes a charge of

$$\frac{w_1(f)}{k} = \frac{k - 3}{k}$$

to each of its k vertices. After this step the charge of each face is 0. Denote by $w_2(\cdot)$ the charge of the vertices after Step 2.

For a vertex $v \in \mathcal{A}(L)$ denote by K_v the 4-multiset of the sizes of the faces in $Z(v)$.

Proposition 2. *Let v be a vertex in $\mathcal{A}(L)$. Then $w_2(v) < 0$ if and only if K_v is one of the following: $\{3, 3, 4, 8\}$, $\{3, 3, 4, 9\}$, $\{3, 3, 4, 10\}$, $\{3, 3, 4, 11\}$ or $\{3, 3, 5, 7\}$.*

Proof. If we exclude the case $n \leq 3$ (for which Theorem 2 is trivial), two faces of size 3 cannot share an edge. Hence there are at most two faces of size 3 in $Z(v)$. In addition, by (7), $\sum_{f \in Z(v)} |f| \geq 18$. Therefore the 4-multiset K_v satisfies the conditions of Lemma 1. We deduce that $\sum_{f \in Z(v)} \frac{|f| - 3}{|f|} < 1$ if and only if K_v is one of the five 4-multisets listed in the proposition. The result follows now since

$$w_2(v) = -1 + \sum_{f \in Z(v)} \frac{|f| - 3}{|f|}.$$

□

We say that two vertices v and u are *neighbors* if $\{v, u\}$ is an edge in $\mathcal{A}(L)$.

For a vertex $u \in \mathcal{A}(L)$ such that $w_2(u) \geq 0$, denote by V_u^- the set of all vertices v in $\mathcal{A}(L)$ with the following two properties:

1. $w_2(v) < 0$, and
2. v and u are neighbors, or v and u are opposite vertices in a face of size 4.

Step 3 [discharging vertices with positive charge]: A vertex $u \in L$ such that $w_2(u) \geq 0$ and $V_u^- \neq \emptyset$ contributes a charge of $\frac{w_2(u)}{|V_u^-|}$ to each one of the vertices in V_u^- . Denote by $w_3(\cdot)$ the charge of the vertices after Step 3. The next proposition completes the proof of Theorem 2:

Proposition 3. *For every vertex $v \in \mathcal{A}(L)$, $w_3(v) \geq 0$.*

Proof. Clearly, $w_2(v) \geq 0$ implies $w_3(v) \geq 0$. We show that $w_3(v) \geq 0$ also in the case $w_2(v) < 0$. Let $v \in \mathcal{A}(L)$ be a vertex such that $w_2(v) < 0$. By Proposition 2, K_v is one of the following: $\{3, 3, 4, 8\}$, $\{3, 3, 4, 9\}$, $\{3, 3, 4, 10\}$, $\{3, 3, 4, 11\}$ or $\{3, 3, 5, 7\}$. We split into two cases:

Case 1. $K_v = \{3, 3, 4, k\}$ for some $8 \leq k \leq 11$. In this case

$$w_2(v) \geq -1 + \frac{1}{4} + \frac{5}{8} = -\frac{1}{8}.$$

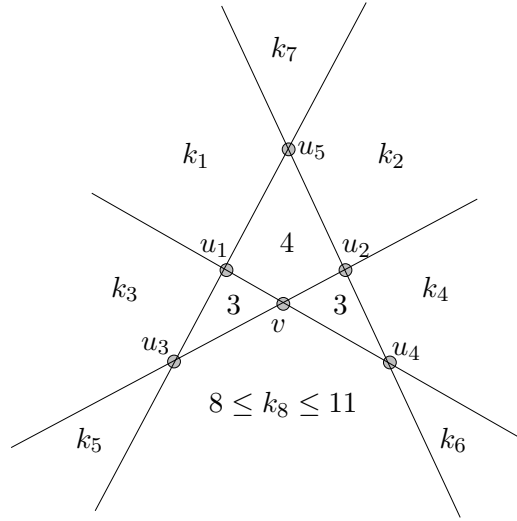


Figure 1: The local neighborhood of v in Case 1.

Figure 1 illustrates the local neighborhood of v , i.e., k_i ($i = 1, \dots, 8$) denote the sizes of the faces f_i in this neighborhood, and u_i ($i = 1, \dots, 5$) denote the vertices. Note that $k_3 \geq 4$ and $k_4 \geq 4$ because f_3 and f_4 share an edge with a face of size 3. Moreover, $k_1 \geq 4$. Indeed, if $k_1 = 3$ then f_1 is supported by the antipodal vertex of u_4 , hence L contains only 4 great circles which is a contradiction to $k_8 \geq 8$ (see Figure 2). A symmetric argument yields $k_2 \geq 4$. Therefore, by Proposition 2, $w_2(u_1) \geq 0$, $w_2(u_2) \geq 0$ and $w_2(u_5) \geq 0$, as the zone of each contains at most one face of size 3. Observe that u_1, u_2, u_3, u_4 are neighbors of v , and that u_5 and v are opposite vertices in a face of size 4. Therefore, for every $1 \leq i \leq 5$, if $w_2(u_i) \geq 0$ then $v \in V_{u_i}^-$.

We shall now consider separately 4 possible subcases:

Subcase 1.1. $w_2(u_3) < 0$ and $w_2(u_4) < 0$. By Proposition 2, $k_5 = k_6 = 3$ and $k_3 = k_4 = 4$. Hence, by (7), we have $k_1 \geq 7$ and $k_2 \geq 7$. Therefore,

$$w_2(u_5) \geq -1 + \frac{1}{4} + \frac{4}{7} + \frac{4}{7} = \frac{11}{28}.$$

We claim that $|V_{u_5}^-| \leq 3$. Indeed, since $v \in V_{u_5}^-$ we have to show that there are at most two more vertices in $V_{u_5}^-$. We have $w_2(u_1), w_2(u_2) \geq 0$, and hence $u_1, u_2 \notin V_{u_5}^-$. Moreover, $k_1, k_2 \geq 7$,

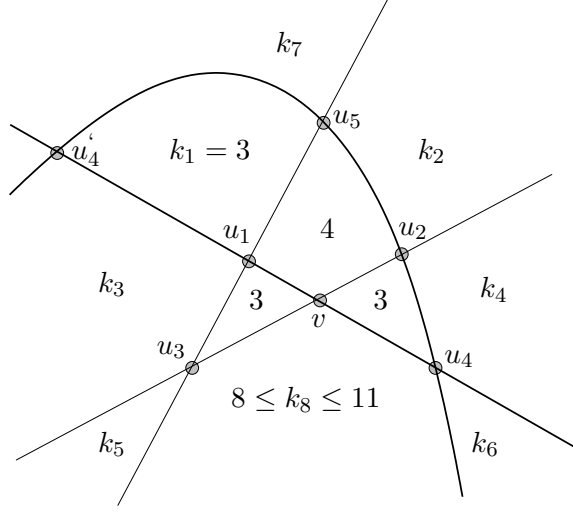


Figure 2: If $k_1 = 3$ the antipodal vertex of u_4 supports f_1 .

so $k_1, k_2 \neq 4$, and thus none of the vertices in f_1 and f_2 is an opposite vertex to u_5 in a face of size 4. There are three options: If $k_7 = 3$ then at most two vertices of f_1 are in $V_{u_5}^-$ (the two neighbors of u_5), and if $k_7 = 4$ then at most one vertex of f_1 is in $V_{u_5}^-$ (the opposite vertex to u_5). In both cases the neighbors of u_5 have a positive weight by Proposition 2, and hence they are not in $V_{u_5}^-$. Finally, if $k_7 \geq 5$ none of the vertices of f_1 is in $V_{u_5}^-$, and the claim is proved (see Figure 3).

Since $v \in V_{u_5}^-$, we have

$$w_3(v) \geq w_2(v) + \frac{w_2(u_5)}{|V_{u_5}^-|} \geq -\frac{1}{8} + \frac{1}{3} \cdot \frac{11}{28} = \frac{1}{168} > 0,$$

as claimed.

Subcase 1.2. $w_2(u_3) < 0$ and $w_2(u_4) \geq 0$. By Proposition 2, $k_5 = 3$ and $k_3 = 4$, and hence by (7), $k_1 \geq 7$. Similar arguments to those in Subcase 1.1 yield $|V_{u_1}^-| \leq 3$, $|V_{u_2}^-| \leq 2$, $|V_{u_4}^-| \leq 3$ and $|V_{u_5}^-| \leq 3$. Since $v \in V_{u_1}^- \cap V_{u_2}^- \cap V_{u_4}^- \cap V_{u_5}^-$, we have

$$\begin{aligned} w_3(v) &= w_2(v) + \frac{w_2(u_1)}{|V_{u_1}^-|} + \frac{w_2(u_2)}{|V_{u_2}^-|} + \frac{w_2(u_4)}{|V_{u_4}^-|} + \frac{w_2(u_5)}{|V_{u_5}^-|} \\ &\geq -\frac{1}{8} + \frac{1}{3} \left(-1 + \frac{1}{4} + \frac{1}{4} + \frac{4}{7} \right) + \frac{1}{2} \left(-1 + \frac{1}{4} + \frac{k_2-3}{k_2} + \frac{k_4-3}{k_4} \right) \\ &\quad + \frac{1}{3} \left(-1 + \frac{5}{8} + \frac{k_4-3}{k_4} + \frac{k_6-3}{k_6} \right) + \frac{1}{3} \left(-1 + \frac{1}{4} + \frac{4}{7} + \frac{k_2-3}{k_2} + \frac{k_7-3}{k_7} \right) \\ &= -\frac{111}{168} + \frac{5}{6} \left(\frac{k_2-3}{k_2} + \frac{k_4-3}{k_4} \right) + \frac{1}{3} \left(\frac{k_6-3}{k_6} + \frac{k_7-3}{k_7} \right). \end{aligned} \tag{9}$$

By (7) $k_2 + k_4 \geq 11$. Therefore the right hand side of (9) is minimal when $k_6 = k_7 = 3$ and $\{k_2, k_4\} = \{4, 7\}$. We conclude that

$$w_3(v) \geq -\frac{111}{168} + \frac{5}{6} \left(\frac{1}{4} + \frac{4}{7} \right) = \frac{1}{42} > 0,$$

as claimed.

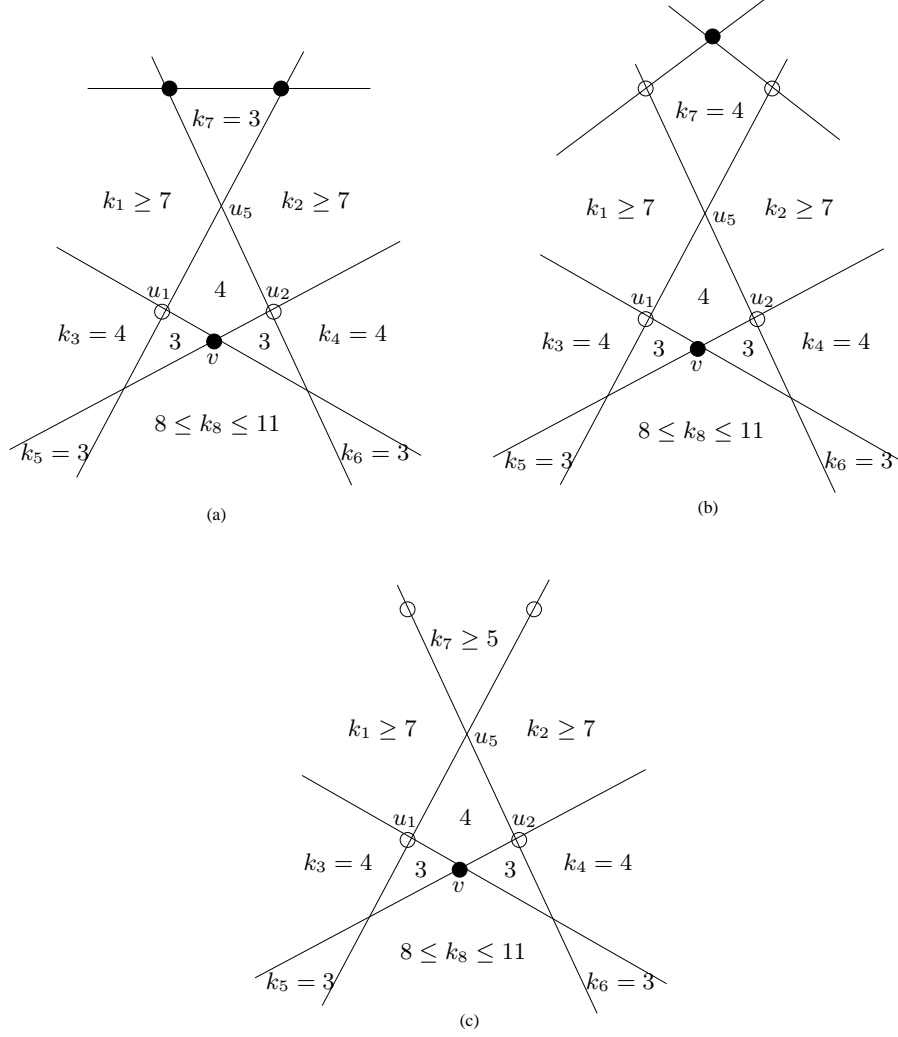


Figure 3: Three different possible scenarios in Subcase 1.1, in which $|V_{u_5}^-| \leq 3$. Vertices with $w_2(v) \geq 0$ are marked by empty circles, while vertices v which might have $w_2(v) < 0$ are marked by full black circles.

Subcase 1.3. $w_2(u_3) \geq 0$ and $w_2(u_4) < 0$. This subcase is symmetric to Subcase 1.2.

Subcase 1.4. $w_2(u_3) \geq 0$ and $w_2(u_4) \geq 0$. Here $|V_{u_1}^-| \leq 2$, $|V_{u_2}^-| \leq 2$, $|V_{u_3}^-| \leq 3$, $|V_{u_4}^-| \leq 3$, and

$|V_{u_5}^-| \leq 3$. Since $v \in \bigcap_{i=1}^5 V_{u_i}^-$, we have

$$\begin{aligned}
w_3(v) &= w_2(v) + \frac{w_2(u_1)}{|V_{u_1}^-|} + \frac{w_2(u_2)}{|V_{u_2}^-|} + \frac{w_2(u_3)}{|V_{u_3}^-|} + \frac{w_2(u_4)}{|V_{u_4}^-|} + \frac{w_2(u_5)}{|V_{u_5}^-|} \\
&\geq -\frac{1}{8} + \frac{1}{2} \left(-1 + \frac{1}{4} + \frac{k_1-3}{k_1} + \frac{k_3-3}{k_3} \right) + \frac{1}{2} \left(-1 + \frac{1}{4} + \frac{k_2-3}{k_2} + \frac{k_4-3}{k_4} \right) \\
&\quad + \frac{1}{3} \left(-1 + \frac{5}{8} + \frac{k_3-3}{k_3} + \frac{k_5-3}{k_5} \right) + \frac{1}{3} \left(-1 + \frac{5}{8} + \frac{k_4-3}{k_4} + \frac{k_6-3}{k_6} \right) \\
&\quad + \frac{1}{3} \left(-1 + \frac{1}{4} + \frac{k_1-3}{k_1} + \frac{k_2-3}{k_2} + \frac{k_7-3}{k_7} \right) \\
&= -\frac{11}{8} + \frac{5}{6} \left(\frac{k_1-3}{k_1} + \frac{k_2-3}{k_2} + \frac{k_3-3}{k_3} + \frac{k_4-3}{k_4} \right) + \frac{1}{3} \left(\frac{k_5-3}{k_5} + \frac{k_6-3}{k_6} + \frac{k_7-3}{k_7} \right). \quad (10)
\end{aligned}$$

Consider the expression

$$\frac{k_1-3}{k_1} + \frac{k_2-3}{k_2} + \frac{k_3-3}{k_3} + \frac{k_4-3}{k_4}. \quad (11)$$

Recall that from (7), $k_1 + k_3 \geq 11$ and $k_2 + k_4 \geq 11$, and that $k_i \geq 4$ for $i = 1, 2, 3, 4$. Thus, if at least one of k_5, k_6, k_7 is greater than or equal to 4, (11) attains its minimum when $\{k_1, k_3\} = \{k_2, k_4\} = \{4, 7\}$. Therefore, in (10),

$$w_3(v) \geq -\frac{11}{8} + \frac{5}{6} \left(\frac{1}{4} + \frac{4}{7} + \frac{1}{4} + \frac{4}{7} \right) + \frac{1}{3} \cdot \frac{1}{4} = \frac{13}{168} > 0,$$

as claimed. Now, assume $k_5 = k_6 = k_7 = 3$. As $w_2(u_3) \geq 0$ and $w_2(u_4) \geq 0$, we have that $k_3 \geq 5$ and $k_4 \geq 5$ by Proposition 2. Moreover, since $k_7 = 3$, (7) implies also that $k_1 + k_2 \geq 11$. Thus, $\{k_1, k_2, k_3, k_4\}$ has to be one of the following: $\{k_1 \geq 4, k_2 \geq 7, k_3 \geq 7, k_4 \geq 5\}$, $\{k_1 \geq 5, k_2 \geq 6, k_3 \geq 6, k_4 \geq 5\}$, $\{k_1 \geq 6, k_2 \geq 5, k_3 \geq 5, k_4 \geq 6\}$, $\{k_1 \geq 6, k_2 \geq 6, k_3 \geq 5, k_4 \geq 5\}$, or $\{k_1 \geq 7, k_2 \geq 4, k_3 \geq 5, k_4 \geq 7\}$. A direct calculation shows that in each one of these options (11) is strictly greater than $\frac{33}{20}$. Therefore, in (10),

$$w_3(v) > -\frac{11}{8} + \frac{5}{6} \cdot \frac{33}{20} = 0,$$

as claimed.

This completes the proof in Case 1.

Case 2. $K_v = \{3, 3, 5, 7\}$. In this case

$$w_2(v) = -1 + \frac{2}{5} + \frac{4}{7} = -\frac{1}{35}.$$

Figure 4 illustrates the local neighborhood of v , i.e., k_i ($i = 1, \dots, 6$) denote the sizes of the faces f_i in this neighborhood, and u_i ($i = 1, \dots, 4$) denote the vertices. Observe that u_1, u_2, u_3, u_4 are neighbors of v . Therefore, for every $1 \leq i \leq 4$, if $w_2(u_i) \geq 0$ then $v \in V_{u_i}^-$.

We shall show that the charge that v receives in Step 3 from the neighbors to its left (u_1 and/or u_3) is greater than $\frac{1}{70}$. Symmetric arguments will show that v also receives such a charge from the vertices to its right (u_2 and/or u_4). Therefore, we shall get that

$$w_3(v) > w_2(v) + 2 \cdot \frac{1}{70} = -\frac{1}{35} + \frac{1}{35} = 0,$$

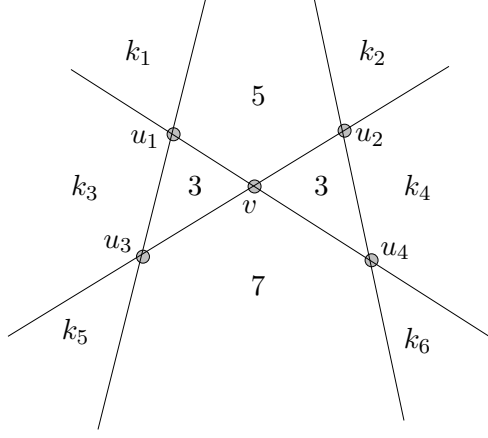


Figure 4: The local neighborhood of v in Case 2.

as required.

To this end we consider 3 possible subcases:

Subcase 2.1. $w_2(u_1) < 0$ and $w_2(u_3) \geq 0$. By Proposition 2, $k_1 = 3$ and $k_3 = 7$. Hence,

$$w_2(u_3) \geq -1 + \frac{4}{7} + \frac{4}{7} = \frac{1}{7}.$$

Moreover, $|V_{u_3}^-| \leq 4$ and $v \in V_{u_3}^-$. Therefore in Step 3 the charge that u_3 contributes to v is

$$\frac{w_2(u_3)}{|V_{u_3}^-|} \geq \frac{1}{4} \cdot \frac{1}{7} = \frac{1}{28} > \frac{1}{70}.$$

Subcase 2.2. $w_2(u_1) \geq 0$ and $w_2(u_3) < 0$. By Proposition 2, $k_3 = 5$ and $k_5 = 3$, hence (7) implies $k_1 \geq 5$. Thus,

$$w_2(u_1) \geq -1 + \frac{2}{5} + \frac{2}{5} + \frac{2}{5} = \frac{1}{5}.$$

Observe that $|V_{u_1}^-| \leq 4$ and that $v \in V_{u_1}^-$. Therefore in Step 3 u_1 contributes to v a charge of

$$\frac{w_2(u_1)}{|V_{u_1}^-|} \geq \frac{1}{4} \cdot \frac{1}{5} = \frac{1}{20} > \frac{1}{70},$$

as well.

Subcase 2.3. $w_2(u_1) \geq 0$ and $w_2(u_3) \geq 0$. We first claim that $|V_{u_1}^-| \leq 3$. Indeed, since $v \in V_{u_1}^-$ we have to show that there are at most two more vertices in $V_{u_1}^-$. We have $w_2(u_3) \geq 0$, thus $u_3 \notin V_{u_1}^-$. Consider three options: If $k_1 = 3$ then by (7) and Proposition 2, $k_3 \geq 8$, and therefore there are at most two more vertices in $V_{u_1}^-$ (the two neighbors of u_1 that belong to f_1). If $k_1 = 4$ then by Proposition 2, $k_3 \geq 6$, and hence there is at most one more vertex in $V_{u_1}^-$ (the opposite vertex to u_1 in f_1). Finally, consider the option $k_1 \geq 5$. Note that $k_3 \geq 4$ since f_3 shares an edge with a face of size 3. Therefore f_1 does not contribute any vertex to $V_{u_1}^-$. Thus, there is at most one more vertex in $V_{u_1}^-$ (the opposite vertex to u_1 in f_3 , in case $k_4 = 4$). This proves the claim, and similar arguments yield $|V_{u_3}^-| \leq 3$, as well (see Figure 5).

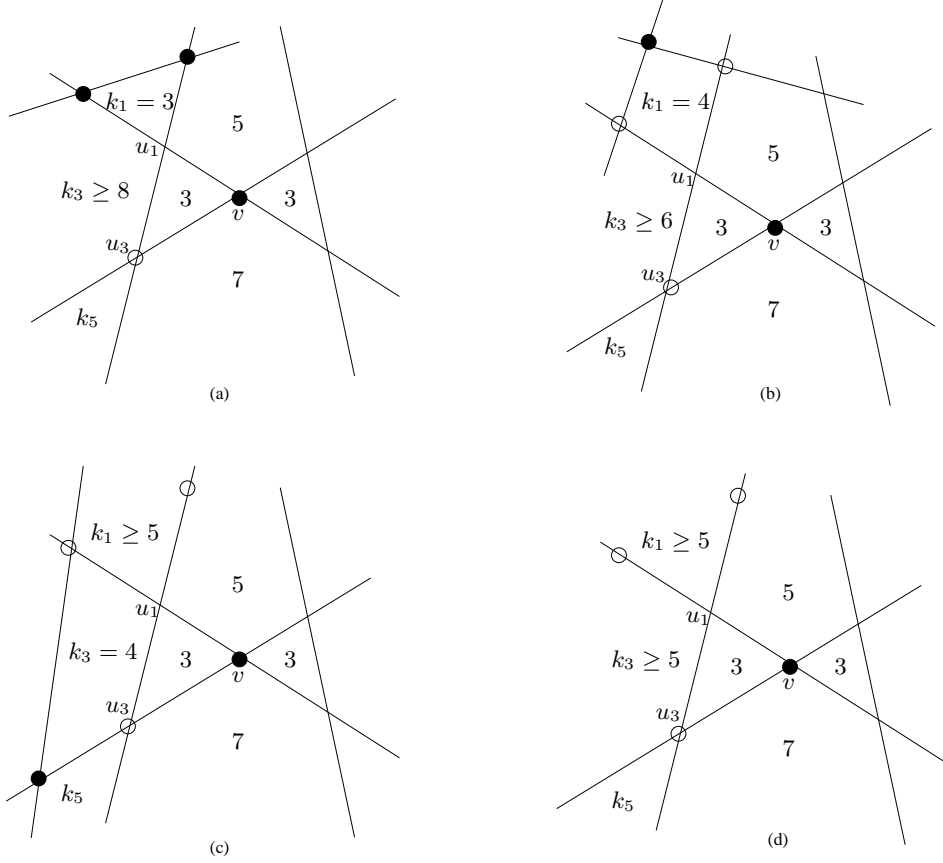


Figure 5: Four different possible scenarios in Subcase 2.3, in which $|V_{u_1}^-| \leq 3$. Vertices with $w_2(v) \geq 0$ are marked by empty circles, while vertices v which might have $w_2(v) < 0$ are marked by full black circles.

Next, we have $v \in V_{u_1}^- \cap V_{u_3}^-$, and therefore in Step 3 u_1 and u_3 contribute to v an overall charge of

$$\begin{aligned} \frac{w_2(u_1)}{|V_{u_1}^-|} + \frac{w_2(u_3)}{|V_{u_3}^-|} &\geq \frac{1}{3} \left(-1 + \frac{2}{5} + \frac{k_1 - 3}{k_1} + \frac{k_3 - 3}{k_3} \right) + \frac{1}{3} \left(-1 + \frac{4}{7} + \frac{k_3 - 3}{k_3} + \frac{k_5 - 3}{k_5} \right) \\ &= -\frac{12}{35} + \frac{2}{3} \left(\frac{k_3 - 3}{k_3} \right) + \frac{1}{3} \left(\frac{k_1 - 3}{k_1} + \frac{k_5 - 3}{k_5} \right) \end{aligned} \quad (12)$$

Let us show that the right hand side of (12) is greater than $\frac{1}{70}$. Note that $k_3 \geq 4$, since f_3 shares an edge with a face of size 3. There are 4 possible options:

1. If $k_3 = 4$ then by (7), $k_1 \geq 6$ and $k_5 \geq 4$. Therefore the right hand side of (12) is at least

$$-\frac{12}{35} + \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \left(\frac{1}{2} + \frac{1}{4} \right) = \frac{31}{420} > \frac{1}{70}.$$

2. If $k_3 = 5$ then (7) implies $k_1 \geq 5$ and by Proposition 2, $k_5 \geq 4$. Thus, the right hand side of (12) is at least

$$-\frac{12}{35} + \frac{2}{3} \cdot \frac{2}{5} + \frac{1}{3} \left(\frac{2}{5} + \frac{1}{4} \right) = \frac{59}{420} > \frac{1}{70}.$$

3. If $k_3 = 6$, we have from (7) that $k_1 \geq 4$. Here the right hand side of (12) is at least

$$-\frac{12}{35} + \frac{2}{3} \cdot \frac{3}{6} + \frac{1}{3} \cdot \frac{1}{4} = \frac{31}{420} > \frac{1}{70}.$$

4. Finally, if $k_3 \geq 7$, the right hand side of (12) is at least

$$-\frac{12}{35} + \frac{2}{3} \cdot \frac{4}{7} = \frac{4}{105} > \frac{1}{70}.$$

To complete the proof in Case 2, note that the subcase where both $w_2(u_1) < 0$ and $w_2(u_3) < 0$ is not possible, since by Proposition 2 it yields $5 = k_3 = 7$, which is absurd.

□

This concludes the proof of Theorem 2.

4 The upper bound of 5 on $C(L)$ is tight

In Figure 6 we give an example of a set L of 10 lines in the real projective plane in general position, such that every vertex $v \in \mathcal{A}(L)$ has $C(v) \geq 5$. The planar arrangement $\mathcal{A}(L)$ has 45 vertices, 30 faces of size 3, 6 faces of size 5 and 10 faces of size 6. There are two types of vertices in $\mathcal{A}(L)$: 30 vertices v with $K_v = \{3, 3, 5, 6\}$ for which $C(v) = 5$, and 15 vertices v with $K_v = \{3, 3, 6, 6\}$ for which $C(v) = 6$. This example shows that the upper bound of 5 on $C(L)$ given in Theorem 2 is tight.

Remark. The example in Figure 6 is the only example we found of an arrangement L with $C(L) = 5$. It would be interesting to know whether there are general such examples, or that the upper bound given in Theorem 1 can actually be improved for large enough n .

Acknowledgments. The author is grateful to Rom Pinchasi for many valuable discussions.

References

- [1] E. Ackerman, K. Buchin, C. Knauer, R. Pinchasi, and G. Rote, There are not too many Magic Configurations, *Disc. and Comp. Geom.* 39, 2008, 3–16.
- [2] M. Bern, D. Eppstein, P. Plassmann and F. Yao, Horizon theorems for lines and polygons, in *Discrete and Computational Geometry: Papers from the DIMACS Special Year* (J. E. Goodman, R. Pollack and W. L. Steiger, editors), AMS, Providence, RI, 1991, pp. 4566.
- [3] B. Chazelle, L. Guibas and D. T. Lee, The power of geometric duality, *BIT* 25, 1985, pp. 76-90.
- [4] H. Edelsbrunner, J. O'Rourke and R. Seidel, Constructing arrangements of lines and hyperplanes with applications, in *SIAM J. Comput.* 15, 1986, pp. 341-363.
- [5] R. Radoičić, G. Tóth, The discharging method in combinatorial geometry and its application to Pach-Sharir conjecture on intersection graphs, in: J.E. Goodman, J. Pach, J. Pollack (Eds.), *Proceedings of the Joint Summer Research Conference on Discrete*

and Computational Geometry, Contemporary Mathematics, AMS, Providence, 2008, pp. 319-342.

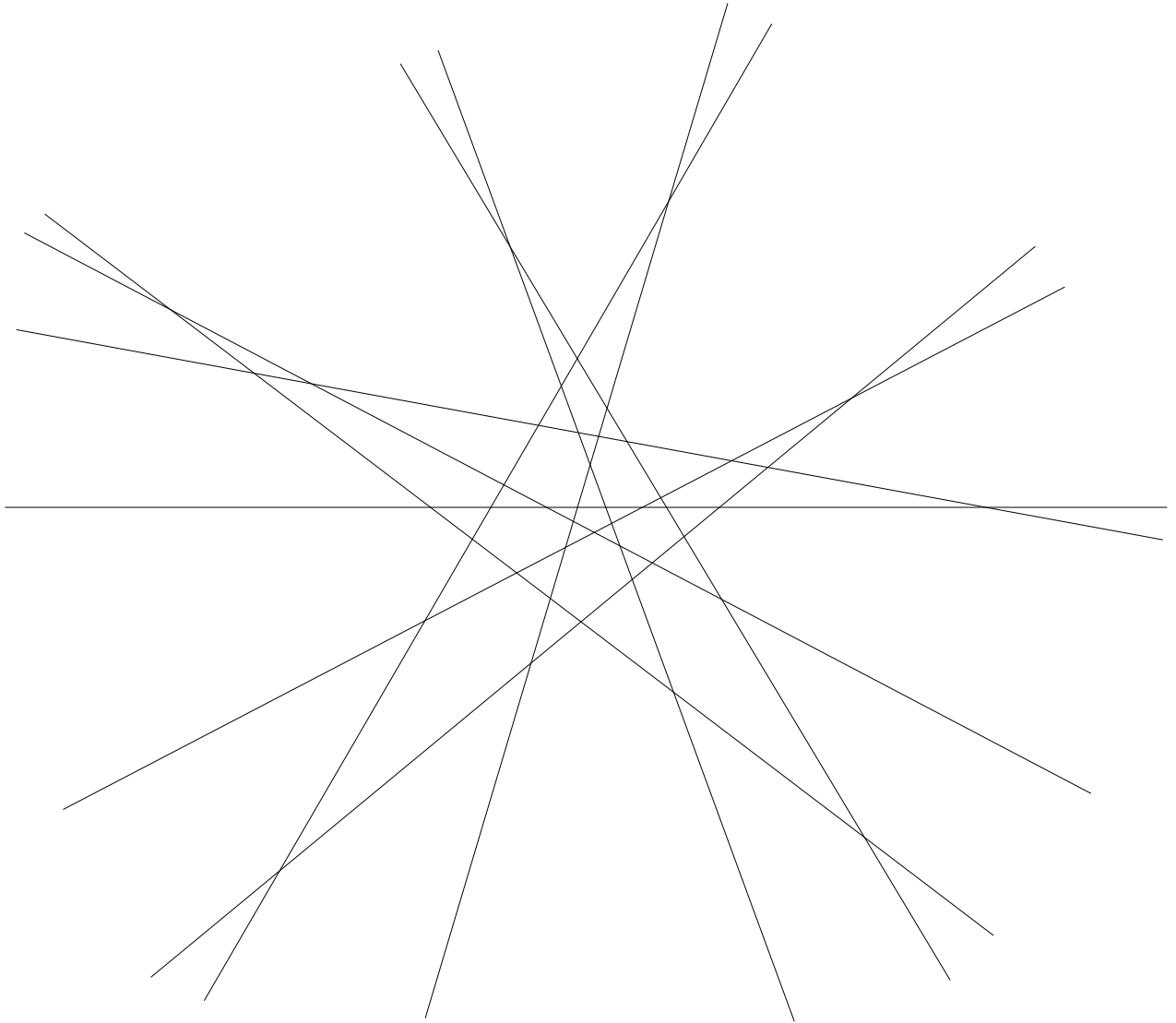


Figure 6: A set L of 10 lines such that every vertex $v \in \mathcal{A}(L)$ has $C(v) \geq 5$.